Eddy currents and corner singularities

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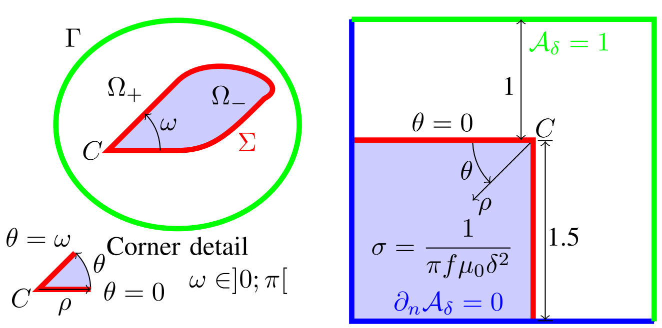
***Abstract*—The first terms of a multiscale expansion are introduced to tackle a magneto-harmonic problem in a bidimensional setting where the conducting medium is non-magnetic and has a corner singularity. The heuristics of the method are given and numerical computations illustrate the obtained accuracy.**

I. Introduction

Electrothermics applications require a precise knowledge of the Joule power density. Skin effect combined with corner singularities is an obstacle to reach this precision. Here, we introduce a method to tackle a magneto-harmonic problem in 2D where the conducting medium is *non-magnetic* and has a *corner singularity*. More precisely, denote by Ω_- the bounded domain corresponding to the conducting medium, and by Ω_+ the surrounding dielectric medium (see Fig. 1(a)). The domain Ω with boundary Γ is defined by $\Omega = \Omega_- \cup \Omega_+ \cup \Sigma$, where Σ is the boundary of Ω_-. For simplicity's sake, we assume that:

(H1) Σ has only one geometric singularity, and we denote by C this corner. The angle of the corner (from the conducting material, see Fig. 1(a)) is denoted by ω.

(H2) the current source term J is located in Ω_+ and it vanishes in a neighborhood of C.



(a) Model domain for the heuristics. (b) L-shape dielectric domain and boundary conditions for the example.

Fig. 1. Geometry of the problems considered.

Throughout the paper ρ denotes the corner distance and θ is the angular variable (see Fig. 1). Moreover the notations $[u]_\Sigma = u^+|_\Sigma - u^-|_\Sigma$ and $\partial_n = n \cdot \nabla$ are used, n being the normal to Σ inwardly directed from Ω_+ to Ω_-. The skin depth $\delta = \sqrt{1/(\pi f \sigma \mu_0)}$ is supposed to be small compared to the characteristic length of the domain. In the expression of δ, f is the frequency of the source term, σ is the conductivity, and μ_0 is the vacuum magnetic permeability. The magnetic vector potential $\mathcal{A}_\delta$ (reduced to one scalar component in 2D) satisfies

$$\left\{ \begin{array}{rcll} -\Delta \mathcal{A}_\delta^+ &=& \mu_0 J & \text{in} \quad \Omega_+, \\ -\Delta \mathcal{A}_\delta^- + \dfrac{2\mathrm{i}}{\delta^2}\mathcal{A}_\delta^- &=& 0 & \text{in} \quad \Omega_-, \\ [\mathcal{A}_\delta]_\Sigma &=& 0 & \text{on} \quad \Sigma, \\ [\partial_n \mathcal{A}_\delta]_\Sigma &=& 0 & \text{on} \quad \Sigma, \\ \mathcal{A}_\delta^+ &=& 0 & \text{on} \quad \Gamma. \end{array} \right. \tag{1}$$

Denote by $\mathcal{A}_0$ the potential in the perfectly conducting case:

$$\left\{ \begin{array}{rcll} -\Delta \mathcal{A}_0^+ &=& \mu_0 J & \text{in} \quad \Omega_+, \\ \mathcal{A}_0^+ &=& 0 & \text{on} \quad \Sigma, \\ \mathcal{A}_0^+ &=& 0 & \text{on} \quad \Gamma, \end{array} \right. \qquad \mathcal{A}_0^- = 0 \quad \text{in} \quad \Omega_-. \tag{2}$$

It is intuitive that $\mathcal{A}_0^+$ approximates $\mathcal{A}_\delta$ in the dielectric medium. Moreover, it can be proved for a regular interface Σ that the "power norm" [1] of the error $\mathcal{A}_\delta - \mathcal{A}_0$ is of order δ [2]. This accuracy is no more valid for a corner singularity. Our aim is to propose a rigorous method to recover the order δ by adding an appropriate correction in the neighborhood of the corner. Note that Yuferev *et al.* in [3] have considered a similar problem using a formal approach of transmitted singularities. Their work aimed at "correcting" the method proposed by Deeley [4]. However we are confident that the heuristics of [3] lead to non-relevant results. This is detailed in Section III.

Here, we present the heuristics of the treatment of the singularity, that lead to the accurate approximation of $\mathcal{A}_\delta$ as δ goes to zero, and we conclude by numerical experiments.

II. Heuristics of the expansion

Let first note the two following remarks:

- similarly to the regular case, $\mathcal{A}_0$ defined by (2) is the solution of the limit problem of (1) as δ goes to zero. Hence the first term of the expansion should start by $\mathcal{A}_0$.
- since the respective behaviors of $\mathcal{A}_\delta$ and $\mathcal{A}_0$ are different in the corner for any non-zero δ, it seems natural to truncate $\mathcal{A}_0$ in the corner by a function φ which is zero close to the corner and 1 far from this corner. Suppose that we introduce such a smooth radial cut-off function:

$$\varphi(\rho) = \begin{cases} 1, & \text{if } \rho \geqslant d_1 \\ 0, & \text{if } \rho \leqslant d_0 \end{cases}, \quad \text{with } d_0 < d_1, \tag{3}$$

d_0, d_1 being fixed corner distances. If $\varphi \mathcal{A}_0$ is taken as the first term for approximating $\mathcal{A}_\delta$, it will not converge to $\mathcal{A}_0$

as δ goes to zero. If $\varphi(\cdot/\delta)\mathcal{A}_0$ is considered instead, the correct limit is obtained ($\varphi(./\delta)$ function: $t \mapsto \varphi(t/\delta)$).

According to these remarks, consider the problem satisfied by $r_0^\delta = \mathcal{A}_\delta - \varphi(./\delta)\mathcal{A}_0$:

$$-\Delta r_0^\delta = [\Delta; \varphi(./\delta)]\, \mathcal{A}_0^+, \quad \text{in } \Omega_+, \quad r_0^\delta|_\Gamma = 0, \text{ on } \Gamma, \tag{4a}$$

$$-\Delta r_0^\delta + \frac{2\mathrm{i}}{\delta^2} r_0^\delta = 0, \quad \text{in } \Omega_-, \tag{4b}$$

$$\left[r_0^\delta\right]_\Sigma = 0, \; \left[\partial_n r_0^\delta\right]_\Sigma = -\partial_n\left(\varphi(./\delta)\mathcal{A}_0^+\right), \quad \text{on } \Sigma, \tag{4c}$$

where for any couple (ν, u), $[\Delta; \nu]u = \Delta(\nu u) - \nu\Delta u$. Note that assumption (H2) is necessary to obtain (4a).

If we were not to use the cut-off function φ in the corner, therefore the jump $\left[\partial_n r_0^\delta\right]_\Sigma$ would be equal to $-\partial_n \mathcal{A}_0^+|_\Sigma$, which blows up in the corner. Since $[\partial_n \mathcal{A}_\delta]_\Sigma$ identically vanishes in the corner on Σ we would have to compensate this blowing term, which would lead to numerical difficulties. The use of $\varphi(./\delta)$ in (4c) ensures that $\left[\partial_n r_0^\delta\right]_\Sigma$ vanishes in the corner. Solving exactly (4) provides no benefits, but since

$$\mathcal{A}_0^+ \underset{\rho\to 0}{\simeq} a_1 \rho^\alpha \sin(\alpha\theta) = a_1 \mathfrak{s}^\alpha \text{ where } \alpha = \pi/(2\pi - \omega), \tag{5}$$

we guess a correction in the corner region such that the expansion becomes

$$\mathcal{A}_\delta = \varphi\left(\frac{\cdot}{\delta}\right)\mathcal{A}_0 + (1-\varphi) a_1 \delta^\alpha V_\alpha\left(\frac{\cdot}{\delta}\right) + r_\alpha^\delta. \tag{6}$$

In (6), the "profile" term V_α is the solution of a problem in $\mathbb{R}^2$ that is *independent of* $\mathcal{A}_0$ *and* δ while r_α^δ lives in the domain Ω. To determine the problem solved by V_α, from (5) we first replace $\mathcal{A}_0^+$ by $\mathfrak{s}^\alpha$ in (4). Then we use the fact that φ depends only on ρ and that $\partial_n = \pm(1/\rho)\partial_\theta$ near the corner, and we perform the rescaling $X = x/\delta$ ($R = \rho/\delta$). Taking the limit when δ goes to zero (Γ is thus "sent" to the infinite) leads to the "profile" problem satisfied by V_α in $\mathbb{R}^2$, which is divided into two infinite sectors S_+ and S_- (remember that $X = (R\cos(\theta), R\sin(\theta))$ with $R > 0$):

$$-\Delta_X V_\alpha = [\Delta_X; \varphi]\, \mathfrak{s}^\alpha, \text{ in } S_+ = \{X : \theta \in (\omega, 2\pi)\}, \tag{7a}$$

$$-\Delta_X V_\alpha + 2\mathrm{i} V_\alpha = 0, \text{ in } S_- = \{X : \theta \in (0, \omega)\}, \tag{7b}$$

$$V_\alpha \to_{|X|\to+\infty} 0, \tag{7c}$$

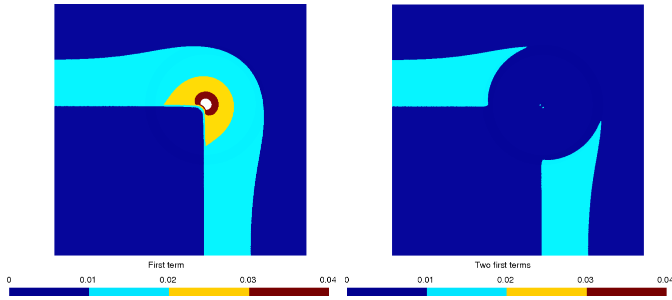
with the transmission conditions on $\mathcal{G} = \{X : \theta = 0, \omega\}$:

$$[V_\alpha]_\mathcal{G} = 0, \quad [\partial_n V_\alpha]_\mathcal{G} = \alpha\varphi R^{\alpha-1}. \tag{7d}$$

Capturing the singularity of the domain in a profile term is quite natural and has to be linked up similarly to [5], [6]. The theoretical proof that r_α^δ is of order δ needs more than two pages, and will be presented in a forthcoming paper.

III. Numerical results

The domain presented in Fig. 1(b) is considered for numerical purpose. The errors $|r_0^\delta|$ and $|r_\alpha^\delta|$ are plotted respectively in Fig. 2(a) and 2(b). The terms $\mathcal{A}_\delta$, a_1, $\mathcal{A}_0$ and V_α are computed by using the finite element method as in [6] where an electrostatic problem on a geometry with a rounded corner is considered. On both figures, the same color scale is used except the white area around the corner in Fig. 2(a) where the error is higher (between 0.04 and 0.14). Fig. 2(b) shows the profile correction (7): the highest error lies now in the regular part of the interface Σ, for which correction is known [2].



(a) $|r_0^\delta|$. (b) $|r_\alpha^\delta|$.

Fig. 2. Modulus of the errors between the solution and the two first orders of (6) for $\delta = 0.025$. The distances of (3) are $d_0 = 1$ and $d_1 = 1.2$.

Suppose that $a_1 \neq 0$, which is the worst corner influence, and denote by $Z_s = (1+\mathrm{i})/(\sigma\delta)$ the regular surface impedance. According to the expansion, the surface impedance Z_δ can be approximated close to the corner by:

$$Z_\delta = Z_s \frac{1+\mathrm{i}}{\delta}\frac{\mathcal{A}_\delta}{\partial_n \mathcal{A}_\delta} \underset{\rho\to 0}{\simeq} Z_s(1+\mathrm{i})\frac{V_\alpha(\cdot/\delta)}{(\partial_n V_\alpha)(\cdot/\delta)}, \tag{8}$$

therefore for any σ and f such that δ is small enough, the function $Z_\delta(\delta\cdot)/|Z_s|$ behaves close to zero as $\sqrt{2}\mathrm{i}V_\alpha/(\partial_n V_\alpha)$. These similar behaviors are shown in Fig. 3 where the "impedance" from the profile function is compared to the real impedance for two values of δ, where f and σ are different. According to [3], the surface impedance should blow up like ρ^{-1} for any non-zero δ, which is shown to be false here.

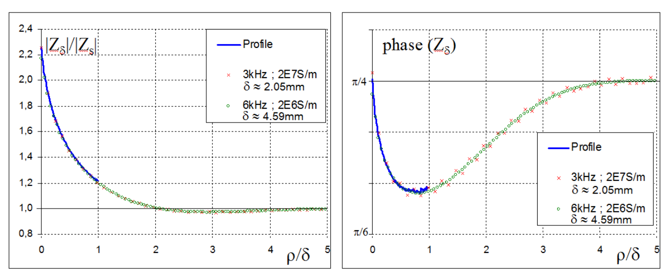


Fig. 3. Behavior of $Z_\delta/|Z_s|$ vs ρ/δ. The domain characteristic length L is here 0.1m, then δ/L is between 2 and 4.6% for the situations considered.

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